**Eigenfunction Expansions**

Let’s go back to the inhomogeneous linear equation. We want to examine a way to solve this equation that’s kind of like the series approach – we’ll write the solution as an infinite sum. But let’s step back and consider a second order Hermitian operator, . A differential operator, , is Hermitian, if for any function on the interval.



Often the Hermiticity of an operator will depend both on the form of the operator and the *boundary conditions* imposed as well; we’ll consider the boundary conditions of the operator as part of the operator itself in a sense. This can be easily extended to the multivariable case - just a double, or triple integral, over a boundary. Now consider ODE`s of the form:



Since H is Hermitian, there will be a denumerably infinite, if a and b are finite, continuously infinite otherwise, set of real eigenvalues, λ, corresponding, to perhaps more than one, eigenfunction which is a solution to the operator equation. In addition the eigenfunctions corresponding to different eigenvalues will be orthogonal with respect to ρ(t) (which shouldn't change sign over the interval; always > 0, or < 0 ). This is because,



And so we can see that eigenfunctions corresponding to different eigenvalues are orthogonal. By related arguments we have that the eigenvalues are necessarily real too. Also, the eigenfunctions will form a complete set over the interval (a,b), which can be allowed to be infinite so that any function, piece-wise continuos, bounded?, can be expanded in terms of them on that interval, and the series will converge in the mean-square sense. What sort of operators are Hermitian...?

**Sturm Liouville Operators**

Consider an ODE operator of the form



If p(t) doesn't change sign - is always greater or less than 0 - and is differentiable in the entire interval, and q(t) is continuous in the interval, and with linear homogenous BC of the form:



Then is Hermitian. This is seen from the following. Let f and g be solutions of the differential equation with boundary conditions. And we will establish Lagrange's identity.



So we have:



What if p(x) is singular at both boundary points? Then what condition would be sufficient to make the linear operator Hermitian? Well, consider again the above. Then a sufficient condition for Hermiticity would be:



If p(t), q(t) don't satisfy these conditions at one of the boundary points - usually p(t) = 0 there, or isn't differentiable there (these boundary points are called singular points) - then the above boundary conditions are too restrictive, and all that should be required at a singular point is that y(t) and y´(t) are both finite at the singular point. If the other boundary point is regular, then it should obey the more restrictive conditions above.

**Solving Linear Inhomogeneous Equations**

So consider the equation



The first step would be to find the eigenfunctions of the L-operator. So we’d want to form the equation:



But is L Hermitian? Not necessarily. So will it even have eigenfunctions? As it turns out, it will, because it can be simply put into Sturm-Louiville form. We just multiply through by the weight function:



(note how its the inverse of the wronskian W(x)). Then we have:



So we see that the functions y are eigenfunctions of a Sturm-Liouville operator with weight function w(x). If the boundary conditions have a constant inhomogeneity, then you can substitute in y = w + mt + b, where m and b are to be determined by homogenizing the boundary conditions. H should still be of the same form. So to solve our inhomogeneous problem, subject to the above boundary conditions by the method of eigenfunction expansions, we first find the eigenfunctions of the differential operator, . Then we recognize that the eigenfunctions will be orthogonal w/r to w(x). And then with our complete set of eigenfunctions we expand our desired solution in terms of the complete set and use the basis set orthonormality to solve for the coefficients.



So we see that the solution is:



And so we have:



We of course recognize G(x,t) as a Green’s function.

So in this method you either have a hermitian differential operator equation (homogeneous, and the boundary conditions are included in its hermiticity), or you can make that way, in which case you find the complete set of eigenfunctions of the operator over the desired interval. You then equate the general solution with a sum over these eigenfunctions. You plug in this solution to the ODE, or PDE, and use the orthoganality condition, along with the inhomogeneity, to solve for the unknown coefficients of the eigenfunctions. Note that if you boundary conditions on your problem make the operator incurably non hermitian, you can still just find the eigenfunctions of the hermitian differential operator (with the necessary BC ) and expand the particular solution in an infinite sum of the eigenfunctions, plug into the ODE, or PDE, and get your particular solution. Then you can just find the general solution to the homogeneous equation and then demand that yh +yp obey the desired BC .